Seepage beneath water retaining structures founded on spatially random soil

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Abstract The effect of stochastic soil permeability on confined seepage occurring beneath water retaining structures has been studied. Random field concepts for the generation of soil permeability properties with a fixed mean, standard deviation and spatial correlation structure, have been combined with finite element methods to perform Monte Carlo simulations of the seepage problem. Analyses have been performed for the case of a dam with two cut-off walls. The results of parametric studies to gauge the effect of the standard deviation and correlation structure of the permeability on the output statistics relating to seepage quantities, exit gradients and uplift pressures are presented. In all cases, comparison is made with results that would be achieved on a deterministic basis. Flow rates and other quantities of interest are shown to be significantly affected by both the standard deviation and the correlation structure of soil permeability.

1 Introduction

The great majority of geotechnical analysis is deterministic in that the soil properties used are assumed to be ‘average’ values. Variations in the soil properties are then accounted for by the use of safety factors which are often rather arbitrarily applied to the computed result.

This ‘average’ approach to the definition of soil properties has tended to be applied, not only to classical soil mechanics calculations, but also to numerical computations using sophisticated numerical techniques such as the finite element method. Properties are usually assigned on the basis of a limited number of laboratory tests. In reality, these properties vary from point to point and can only be determined deterministically through numerous field tests. Since this is both expensive and impractical, random field models can be used to represent the geomaterial. The parameters of these models can be estimated from a limited number of test results.

Mean soil properties are fairly well established, and recently there has been an improvement in the availability of data on second moment statistics (standard deviation and spatial correlation). In large part, the data gathering has been motivated by the availability of random field simulation algorithms and their potential for producing useful results. The increased performance of computers has also enabled more detailed discretisation of boundary value problems, and better modelling of the statistical properties of the input parameters.
The finite element method is an ideal vehicle for modelling materials with a spatial variation in properties. Stochastic finite element analysis has been implemented in a number of areas of geotechnical interest. For example Beacher and Ingra (1981) and Righetti and Harrop-Williams (1988) for stress analysis and settlements of foundations, Ishii and Suzuki (1987) for slope stability and Smith and Freeze (1979a and 1979b) for confined seepage.

Stochastic finite elements can be interpreted in different ways. On the one hand statistical properties can be built into the finite element equations themselves (see e.g. Vanmarcke and Grigoriu 1983), or multiple analyses (Monte Carlo) can be performed, with each analysis stemming from a realisation of the soil properties treated as a multi-dimensional random field. In the present work the latter approach has been used to examine confined seepage, with particular reference to flow under a water retaining structure founded on stochastic soil. While the Monte Carlo approach tends to be computationally intensive, it has the distinct advantage of being able to model highly variable input properties.

2 Confined seepage

In the study of seepage through soils beneath water retaining structures, three important quantities need to be assessed by the designers as shown in Figure 1: seepage quantity, exit gradients and uplift forces.

![Figure 1. The boundary value problem](image)

The classical approach used by Civil Engineers for estimating these quantities involves the use of carefully drawn flow nets (Casagrande 1940, Cedergren 1967, Verruijt 1970). Various alternatives to flow nets are available for solving the seepage problem, however in order to perform quick parametric studies, for example relating to the effect
of cut-off wall length, powerful approximate techniques such as the Method of Fragments (Pavlovsky 1933, Harr 1962, Griffiths 1984) are increasingly employed. The conventional methods are deterministic, in that the soil permeability is assumed to be constant and homogeneous, although anisotropic properties and stratification can be taken into account.

In this paper, a more rational approach to the modelling of soil properties is adopted, whereby the permeability of the soil underlying a structure such as that shown in Figure 1 is assumed to be stochastic, i.e. the soil property in question is assumed to be a ‘random’ field (e.g. Vanmarcke 1984) defined statistically. The best known statistics are the mean and standard deviation, however it is well known that spatial dependencies also exist – the soil properties at two points separated by 1 cm are likely to be more similar than those at two points separated by a metre or a kilometre. This spatial dependence is often characterized by a measure called the ‘scale of fluctuation’, which, loosely speaking, is the distance over which properties show appreciable correlation. In general for a site of fixed size, as the scale of fluctuation increases, the soil properties become more uniform over the site. These statistics will be discussed further in a later section.

The analyses in this paper use a technique called Local Average Subdivision (LAS) to generate realisations of the random permeability fields with given mean, standard deviation and correlation structure. This technique is fully described by Fenton (1990) and Fenton and Vanmarcke (1990). The resulting field of permeabilities is mapped onto a finite element mesh, and potential and stream function boundary conditions are specified. The governing elliptic equation for steady flow (Laplace) leads to a system of linear ‘equilibrium’ equations which are solved for the nodal potential values throughout the mesh using conventional Gaussian elimination.

Only deterministic boundary conditions are considered in this paper, the primary goal being to investigate the effects of randomly varying soil properties on the engineering quantities noted above. The method is nevertheless easily extended to random boundary conditions corresponding to uncertainties in upstream and downstream water levels.

The next two sections give a brief description of the finite element technique and the method by which permeability values are assigned to the mesh. This is followed by a results section in which the statistics of the output quantities relating to flow rate, exit gradients and uplift are discussed.

3 Finite Element Analyses

The steady flow problem is governed in 2-d by Laplace’s equation, in which the dependent variable \( \phi \) is the piezometric head or potential at any point in the Cartesian field \( x-y \).

\[
k_x \frac{\partial^2 \phi}{\partial x^2} + k_y \frac{\partial^2 \phi}{\partial y^2} = 0
\]

where \( k_x \) and \( k_y \) are the permeabilities in the \( x \)– and \( y \)–directions. In the present work and at the element level, the permeability field is assumed to be isotropic \( (k_x = k_y = k) \). While the method discussed herein is simply extended to the anisotropic case (through the generation of a pair of correlated random fields) it was felt that such an extension
is best restricted to a particular site of interest, the complexity introduced to a general
discussion being unwarranted.

Note that equation (3.1) is strictly only valid for constant $k$. In this analysis the
permeability is taken to be constant within each element, its value being given by the
local geometric average of the permeability field within the domain of the element. From
element to element, the value of $k$ will vary, however, reflecting the random nature of
the permeability. This approximation of the permeability field is consistent with the
approximations made in the finite element method and is superior to most traditional
approaches in which the permeability of an element is taken to be simply the permeability
at some point within the element.

A typical finite element mesh used in this study is shown in Figure 2. It contains
1400 elements, and represents a model of 2-d flow beneath a dam which includes two
cut-off walls. The upstream and downstream potential values are fixed at 10 and zero
metres respectively. The cut-off walls are assumed to have zero thickness, and the nodes
along those walls have two potential values corresponding to the right and left sides of
the wall.

![Figure 2. The FE mesh: all elements are 0.2m x 0.2m squares](image)

The finite element code for the solution of Laplace’s equation is broadly similar to
that published by Smith and Griffiths (2004). The element conductivity matrices are
assembled into a global matrix in the usual way, resulting in a system of linear equations in
the ‘unknown’ nodal potential values. The global conductivity relationship after assembly
becomes

$$ K\Phi = Q $$

Once the global conductivity equations are solved leading to nodal potential values
held in $\Phi$, the output quantities relating to flow rates, uplift pressures and exit gradients
are easily deduced. More detail on how these values are obtained will be described in
later sections.

For each boundary value problem considered, multiple solutions were obtained using
successive realisations of the permeability field. The random permeability field is char-
acterized by three parameters defining its first two moments, namely the mean $\mu_k$, the
standard deviation $\sigma_k$ and the scale of fluctuation $\theta_k$. 

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In order to obtain reasonably stable output statistics, it was decided that each parametric combination would be analysed using 1000 realisations.

4 Generation of permeability values

Field measurements of permeability have indicated an approximately log-normal distribution (see e.g. Hoeksma and Kitanidis 1985, Sudicky 1986). The same distribution has therefore been adopted for the simulations presented in this Paper. Essentially, the permeability field is obtained through the transformation

\[ k_i = \exp\{\mu_{\ln k} + \sigma_{\ln k} g_i\} \]  

in which \( k_i \) is the permeability assigned to the \( i^{th} \) element, \( g_i \) is the local average of a standard Gaussian random field, \( g \), over the domain of the \( i^{th} \) element, and \( \mu_{\ln k} \) and \( \sigma_{\ln k} \) are the mean and standard deviation of the logarithm of \( k_i \) (obtained from the ‘target’ mean and standard deviation \( \mu_k \) and \( \sigma_k \)).

The LAS technique renders realisations of the local averages \( g_i \) which are derived from the random field \( g \) having zero mean, unit variance, and a spatial correlation controlled by the scale of fluctuation. As the scale of fluctuation goes to infinity, \( g_i \) becomes equal to \( g_j \) for all elements \( i \) and \( j \) – that is the field of permeabilities tends to become uniform on each realisation. At the other extreme, as the scale of fluctuation goes to zero, \( g_i \) and \( g_j \) become independent for all \( i \neq j \) – the soil permeability changes rapidly from point to point.

In the two dimensional analyses presented in this paper, the scales of fluctuation in the vertical and horizontal directions are taken to be equal (isotropic) for simplicity. Although beyond the scope of this paper, it should be noted that for a layered soil mass the horizontal scale of fluctuation is generally larger than the vertical scale due to the natural stratification of many soil deposits. The 2-d model used herein implies that the out-of-plane scale of fluctuation is infinite – soil properties are constant in this direction – which is equivalent to specifying that the streamlines remain in the plane of the analysis. This is clearly a deficiency of the present model, however it is believed that useful information regarding the variability of flow quantities is still to be gained from the 2-d model.

5 Deterministic Solution

With regard to the seepage problem shown in Figure 2, a deterministic analysis was performed in which the permeability of all elements was assumed to be constant and equal to \( 10^{-5} \) m/s. This value was chosen as it was to be the mean value of subsequent stochastic analyses. Both the potential and the inverse streamline problems were solved, leading to the flow net shown in Figure 3.

All output quantities were computed in non-dimensional form. In the case of the flow rate, the global flow vector \( \mathbf{Q} \) was computed by forming the product of the potentials and the global conductivity matrix from equation (3.2). Assuming no sources or sinks in the flow regime, the only non-zero values in \( \mathbf{Q} \) correspond to those freedoms on the
upstream and downstream boundaries. These values were summed to give the flow rate $Q$ in $m^3/s/m$, leading to a non-dimensional flow rate $\bar{Q}$ defined by

$$\bar{Q} = Q/(\mu_k H)$$

where $\mu_k$ is the (isotropic) mean permeability and $H$ is the total head difference between the up- and downstream sides.

The uplift force on the base of the dam $U$ was computed by integrating the pressure distribution along the base of the dam between the cut-off walls. This quantity was easily deduced from the potential values at the nodes along this line together with a simple numerical integration scheme (Repeated Trapezium Rule). A non-dimensional uplift force $\bar{U}$ was defined:

$$\bar{U} = U/(H\gamma_w L)$$

where $\gamma_w$ is the unit weight of water and $L$ is the distance between the cut-off walls. $\bar{U}$ is the uplift force expressed as a proportion of buoyancy force that would occur if the dam was submerged in water alone.

The exit gradient $i_e$ is the rate of change of head at the exit point closest to the dam at the downstream end. This was calculated using a 4-point backward difference numerical differentiation formula of the form:

$$i_e \approx \frac{1}{6b}(11\phi_0 - 18\phi_{-1} + 9\phi_{-2} - 2\phi_{-3})$$

where the $\phi_i$ values correspond to the piezometric head at the four nodes vertically below the exit point as shown in Figure 4, and $b$ is the constant vertical distance between nodes. It may be noted that the downstream potential head is fixed equal to zero, thus $\phi_0 = 0.0$ m. The use of this four-point formula was arbitrary, and was considered a compromise between the use of very low order formulae which would be too sensitive to ‘random’ fluctuations in the potential, and high order formulae which would involve the use of correspondingly high order interpolation polynomials which would be hard to justify physically.

Referring to Figures 1 and 2, the constants described above were given the following values: $H = 10m$, $\mu_k = 10^{-5}m/s$, $\gamma_w = 9.81kN/m^3$, $L = 6m$, $b = 0.2m$. 

**Figure 3.** Deterministic flow net: $n_f = 5$, $n_d = 20$
and a deterministic analysis using the mesh of Figure 2 led to the following output quantities: \( \bar{Q} = 0.226 \), \( \bar{U} = 0.671 \), \( i_e = 0.688 \).

This value of \( i_e \) would be considered unacceptable in a real design situation, bearing in mind that the critical hydraulic gradient for most soils approximately equals unity. The value of \( i_e \) is proportional to the head difference \( H \) however, which in this case for simplicity and convenience of normalisation has been set to 10 m as shown above.

These results will be compared with output from the stochastic analyses described in the next section.

### 6 Stochastic Analyses

In all the 2-d stochastic analyses that follow, the soil was assumed to be isotropic with a mean permeability of \( \mu_k = 10^{-5} \) m/s. More specifically, the random fields were generated such that the ‘target’ (geometric) mean permeability of each finite element was held constant at \( 10^{-5} \) m/s. Parametric studies were performed relating to the effect of varying the standard deviation \( (\sigma_k) \) and the scale of fluctuation \( (\theta_k) \) of the permeability field. Following 1000 realisations, statistics relating to output quantities \( \bar{Q}, \bar{U} \) and \( i_e \) were calculated.
6.1 Single realisation

Before discussing the results from multiple realisations, an example of what a flow net might look like for a single realisation is given in Figure 5 for permeability statistics \( \sigma_k/\mu_k = 1 \), and \( \theta_k = 1.0 \text{m} \).

In Figure 5, the flow net is superimposed on a 'grey-scale' which indicates the spatial distribution of the permeability values. Dark areas correspond to low permeability and light areas to high permeability. The streamlines clearly try to 'avoid' the low permeability zones, but this is not always possible as some realisations may generate a complete 'blockage' of low permeability material in certain parts of the flow regime. This type of 'blockage' is most likely to occur where the flow route is compressed, such as under a cut-off wall. Flow in these (dark) low permeability zones is characterised by the streamlines moving further apart and the equipotentials moving closer together. Conversely, flow in the (light) high permeability zones is characterised by the equipotentials moving further apart and the streamlines moving closer together.

![Figure 5. Stochastic flow net for a typical realisation](image)

Although local variations in the permeability have an obvious effect on the local paths taken by the water as it flows downstream, globally the stochastic and deterministic flow nets exhibit many similarities. The flow is predominantly in a downstream direction, with the fluid flowing down, under and around the cut-off walls. For this reason the statistics of the output quantities might be expected to be rather insensitive to the geometry of the problem (e.g. length of walls etc.), and qualitatively similar to the properties of a 1-d flow problem.

6.2 Statistics of the potential field

Figures 6 gives contours of the mean and standard deviation of the potential field following 1000 realisations for the case where \( \theta_k = 1.0 \text{m} \). Here we have follow an approach used by Smith and Freeze (1979a, 1979b) who presented the results of a series of numerical experiments on both 1-d and 2-d confined flow problems.

The mean potential values given in Figures 6a are very similar to those obtained in the deterministic analysis summarised in the flow net of Figure 3. The standard deviation of the potentials given in Figures 6b indicate the zones in which the greatest uncertainty...
Figure 6. (a) Contours of mean potential values (contour interval=0.5m), (b) Contours of standard deviation of potential values (contour interval=0.16m), $\sigma_k/\mu_k = 1, \theta_k = 1$ m

exists regarding the potential values. It should be recalled that the up- and down-stream (boundary) potentials are deterministic, so the standard deviation of the potentials on these boundaries equals zero. The greatest values of standard deviation occur in the middle of the flow regime, which in the case considered here represents the zone beneath the dam and between the cut-off walls. The standard deviation is virtually constant in this zone. The statistics of the potential field shown in Figure 6 are closely related to the statistics of the uplift force as will be considered in the next section.

6.3 Parameteric studies

The parametric studies based on the mesh of Figure 2 were designed to show the effect of the permeability’s standard deviation, $\sigma_k$, and scale of fluctuation, $\theta_k$, on the output quantities $\bar{Q}, \bar{U}$ and $i_e$. In all cases the mean permeability, $\mu_k$, was maintained constant at $10^{-5}$ m/s.

Instead of plotting $\sigma_k$ directly, the dimensionless coefficient of variation of permeability was used, and the following values were considered: $\sigma_k/\mu_k = 0.125, 0.25, 0.50, 1, 2, 4, 8, 16.0$ together with scales of fluctuation values given by $\theta_k = 0, 1, 2, 4, 8, \infty$ m.

All permutations of these values were analysed, and the results summarised in Figures 7, 8 and 9 in the form of the logarithm (base 10) of $\sigma_k/\mu_k$ vs. the means and standard deviations of $\bar{Q}, \bar{U}$, and $i_e$, denoted $(\mu_{\bar{Q}}, \sigma_{\bar{Q}})$, $(\mu_{\bar{U}}, \sigma_{\bar{U}})$, and $(\mu_{i_e}, \sigma_{i_e})$ respectively.
Figure 7. Coefficient of variation of permeability plotted against, (a) mean flow rate; (b) standard deviation of flow rate
Flow rate  Figure 7a shows a significant fall in $\mu Q$ as $\sigma_k/\mu_k$ increases for $\theta_k < 8$ m. As the scale of fluctuation approaches infinity, the expected value of $Q$ approaches the constant 0.226. This curve is also shown in Figure 7a, although it should be noted it has been obtained through theory rather than simulation. In agreement with this result, the curve $\theta_k = 8$ m shows a less marked reduction in $\mu Q$ with increasing coefficient of variation $\sigma_k/\mu_k$. However, over typical scales of fluctuation, the effect on average flow rate is slight. The decrease in flow rate as a function of the variability of the soil mass is an important observation from the point of view of design. Traditional design practice may very well be relying on this variability to reduce flow rates on average. It also implies that ensuring higher uniformity in the substrate may be unwarranted unless the mean permeability is known to be substantially reduced and/or the permeability throughout the site is carefully measured.

Figure 7b shows the behaviour of $\sigma Q$ as a function of $\sigma_k/\mu_k$. Of particular note is that $\sigma Q$ reaches a maximum corresponding to $\sigma_k/\mu_k$ in the range 1.0 - 2.0 for $\theta_k \leq 8$ m. Again the theoretical result corresponding to $\theta_k = \infty$ has been plotted on the figure, showing a continuous increase with $\sigma_k/\mu_k$.

In general, it appears that the greatest variability in $Q$ occurs under rather typical conditions: scales of fluctuation between 1 and 4 m and coefficient of variation of permeability of around 1 or 2.

Uplift forces  Figures 8a and 8b show the relationship between uplift force parameters $\mu U$ and $\sigma U$ and input permeability parameters $\sigma_k/\mu_k$ and $\theta_k$. From Figure 8a, $\mu U$ is relatively insensitive to the parametric changes. There is a gradual fall in $\mu U$ as both $\sigma_k/\mu_k$ and $\theta_k$ increase. The greatest reduction being about 10% of the deterministic value of 0.671 when $\sigma_k/\mu_k = 16.0$ and $\theta_k = 8.0$ m. The insensitivity of the uplift force to the permeability input statistics might have been predicted from Figures 3, and 6a in which the contours of (mean) piezometric head are virtually the same in both the deterministic and stochastic analyses.

Figure 8b shows that $\sigma U$ consistently rises as both $\sigma_k/\mu_k$ and $\theta_k$ increase. It is known that in the limit as $\theta_k \to \infty$, $\sigma U \to 0$ since under those conditions, the permeability field becomes completely uniform. Some hint of this increase followed by a decrease is seen from Figure 8b in that the largest increases are for $\theta_k = 0$ to $\theta_k = 1$ while the increase from $\theta_k = 4$ to $\theta_k = 8$ is much smaller.

The actual value of $\sigma U$ for a given set of $\sigma_k/\mu_k$ and $\theta_k$ could easily be deduced from the standard deviation of the potential values. Figure 6b gave contours of the standard deviation of the potential values throughout the flow domain for the particular value $\sigma_k/\mu_k = 1.0$ and $\theta_k = 1.0$ m. In Figure 6b, the potential standard deviation beneath the dam was approximately constant and equal to 0.8 m. After non- dimensionalisation by dividing by $H = 10$ m, these values closely agree with the corresponding values in the graphs of Figure 8b.

The magnitude of the standard deviation of the uplift force given in Figure 8b across the range of parameters considered was not very great. The implication is that this quantity can be estimated with a reasonable degree of confidence. The explanation lies in the fact that the uplift force is calculated using potential values over quite a large number of nodes beneath the dam. This ‘averaging’ process would tend to damp out
Figure 8. Coefficient of variation of permeability plotted against, (a) mean uplift force; (b) standard deviation of uplift force.
fluctuations in the potential values that would be observed on a local scale, resulting in a variance reduction.

Exit gradients  This quantity is based on the first derivative of piezometric head or potential with respect to distance at the exit point closest to the downstream end of the dam. It is well known that in a deterministic approach, the largest value of $i_e$, and hence the most critical, lies at the exit point of the uppermost (and shortest) streamline. While for a single realisation of a stochastic analysis this may not be the case, on average the location of the critical exit gradient is expected to occur at the ‘deterministic’ location.

As $i_e$ is based on a first derivative at a particular location within the mesh (see Figure 4), it can be expected to be the most susceptible to local variations generated by the stochastic approach. In order to average the calculation of $i_e$ over a few nodes, it was decided to use a 4-point (backward) finite difference scheme as given previously in equation (5.3). This is equivalent to fitting a cubic polynomial over the potential values calculated at the four nodes closest to the exit point adjacent to the downstream cut-off wall. The cubic is then differentiated at the required point to estimate $i_e$. Note then that the gradient is estimated by studying the fluctuations over a length of 0.6 m vertically (the elements are 0.2 m by 0.2 m in size). This length will be referred to as the ‘differentiation length’ in the following.

The variation of $\mu_{i_e}$ and $\sigma_{i_e}$, over the range of parameters considered are given in Figures 9a and 9b. The sensitivity of $i_e$ to $\sigma_k/\mu_k$ is clearly demonstrated. In Figure 9a, $\mu_{i_e}$ agrees quite closely with the deterministic value of 0.688 for values of $\sigma_k/\mu_k$ in the range 0.0 to 1.0, but larger values start to show significant instability and divergence. It is interesting to note that for $\theta_k \leq 1$, the tendency is for $\mu_{i_e}$ to fall below the deterministic value of $i_e$ as $\sigma_k/\mu_k$ is increased, whereas for larger values of $\theta_k$ it tends to increase above the deterministic value. The scales 0 and 1 are less than and of the same magnitude as the differentiation length of 0.6 m used to estimate the exit gradient, respectively, while the scales 2, 4, and 8 are substantially greater. If this has some bearing on the divergence phenomena seen in Figure 9a it calls into some question the use of a differentiation length to estimate the derivative at a point. Suffice to say that there may be some conflict between the numerical estimation method and random field theory regarding the exit gradient that needs further investigation.

Figure 9b indicates the relatively large values of $\sigma_{i_e}$, which grow rapidly as $\sigma_k/\mu_k$ is increased. The influence of $\theta_k$ in this case is not so great, with the results corresponding to $\theta_k = 1.0, 2.0, 4.0$ and 8.0 m being quite closely grouped. It is noted that theoretically as $\theta_k \to \infty$ and $\theta_k \to 0$, $\mu_{i_e} \to 0.688$ and $\sigma_{i_e} \to 0$. Their appears to be some evidence of a reduction in $\sigma_{i_e}$ as $\theta_k$ increases, which is in agreement with the theoretical result. For scales of fluctuation negligible relative to the differentiation length, that is $\theta_k = 0$, the variability in $i_e$ is much higher than that for other scales at all but the highest permeability variance. This is perhaps to be expected, since $\theta_k = 0$ yields large fluctuations in permeability within the differentiation length.
Figure 9. Coefficient of variation of permeability plotted against, (a) mean exit gradient; (b) standard deviation of exit gradient.
7 Concluding Remarks

A range of parametric studies have been performed relating to flow beneath a water retaining structure with two cut-off walls founded on a stochastic soil. Random field concepts were used to generate permeability fields having predefined mean, standard deviation and correlation structure. These values were mapped onto a finite element mesh consisting of 1400 elements, and, for each set of parameters, 1000 realisations of the boundary value problem were analysed. In all cases, the ‘target’ mean permeability of each finite element was held constant and parametric studies were performed over a range of values of coefficient of variation and scale of fluctuation.

The three output quantities under scrutiny were the flow rate, the uplift force and the exit gradient; the first two of these being non-dimensionalised for convenience of presentation.

The mean flow rate was found to be relatively insensitive to typical scales of fluctuation, but fell consistently as the variance of the permeability was increased. This observation may be of some importance in the design of such water retaining structures. The standard deviation of the flow rate consistently increased with the scale of fluctuation, but rose and then fell again as the coefficient of variation was increased. These maxima are currently the subject of further investigations by the Authors.

The mean uplift force was rather insensitive to the parametric variations, falling by only about 10% in the worst case. The relatively small variability of uplift force was due to a ‘damping out’ of local variations inherent in the random field by the averaging of potential values over the nodes along the full length of the base of the dam. Nevertheless, the standard deviation of the uplift force rose consistently with increasing scale of fluctuation and coefficient of variation, as was to be expected from the contour plots of the standard deviation of the potential values across the flow domain.

The mean exit gradient was much more sensitive to the statistics of the input field. Being based on a first derivative of piezometric head with respect to length at the exit point, this quantity is highly sensitive to local variations inherent in the potential values generated by the random field. Some local ‘averaging’ was introduced by the use of four-point numerical differentiation formula, however the fluctuation in mean values was still considerable and the standard deviation values high.

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9 Notation

\( b \) \hspace{1cm} \text{vertical spacing between nodes}
\( g \) \hspace{1cm} \text{standard Gaussian random field}
\( g_i \) \hspace{1cm} \text{local average of } g \text{ over the } i^{th} \text{ element}
\( H \) \hspace{1cm} \text{head difference between up- and downstream levels}
\( i_e \) \hspace{1cm} \text{exit hydraulic gradient}
\( k \) \hspace{1cm} \text{permeability}
\( k_i \) \hspace{1cm} \text{permeability assigned to the } i^{th} \text{ element}
\( k_x, k_y \) \hspace{1cm} \text{permeability in } x- \text{ and } y- \text{directions}
\( L \) \hspace{1cm} \text{width of dam}
\( n_f \) \hspace{1cm} \text{number of flow channels}
\( n_d \) \hspace{1cm} \text{number of equipotential drops}
\( Q, \bar{Q} \) \hspace{1cm} \text{flow rate and non-dimensional flow rate}
\( U, \bar{U} \) \hspace{1cm} \text{uplift force and non-dimensional uplift force}
\( x, y \) \hspace{1cm} \text{Cartesian coordinates}
\( \gamma_w \) \hspace{1cm} \text{unit weight of water}
\( \theta_k \) \hspace{1cm} \text{scale of fluctuation of permeability}
\( \mu_k, \sigma_k \) \hspace{1cm} \text{mean and standard deviation of permeability}
\( \mu_{ln k}, \sigma_{ln k} \) \hspace{1cm} \text{mean and standard deviation of } \ln k_i
\( \mu_{i_e}, \sigma_{i_e} \) \hspace{1cm} \text{mean and standard deviation of exit gradient}
\( \mu_{\bar{Q}}, \sigma_{\bar{Q}} \) \hspace{1cm} \text{mean and standard deviation of dimensionless flow rate}
\( \mu_{\bar{U}}, \sigma_{\bar{U}} \) \hspace{1cm} \text{mean and standard deviation of dimensionless uplift force}
\( \sigma_k / \mu_k \) \hspace{1cm} \text{coefficient of variation of permeability}
\( \phi \) \hspace{1cm} \text{potential or piezometric head}
\( \phi_0, \phi_{-1}, \phi_{-2}, \phi_{-3} \) \hspace{1cm} \text{potential values near exit point}

\( K \) \hspace{1cm} \text{global conductivity matrix}
\( Q \) \hspace{1cm} \text{global flow vector}
\( \Phi \) \hspace{1cm} \text{global potential vector}

Bibliography


